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## PROGRESS REPORT.

The Managing Committee is glad to announce that the following gentlemen have kindly consented to be enrolled as Honorary Members of the Society :—

Dr. G. T. Walker, Director-General of Observatories in India, Simla.

Professor G. H. Hardy, New College, Oxford.

Professor G. A. Miller, University of Illinois, .U S. A.

The following gentlemen have been admitted as Members of the Society :—

M. D. Bhat, Esq., I. C. S., Ahmedabad, Guzerat.

Ram Behari Lal, M.A., Professor, St. Stephen's College, Delhi.

V. D. Gokhale, D.Sc., now at Chicago, U. S. A.

Professor D. D. Kapadia, M.A., has become a Life Member of the Society, thereby evincing his genuine interest in and strong attachment to the Society.

In this connection it may be mentioned that the Managing Committee of the Society passed the following resolution in appreciation of the valuable services rendered to the Society by Mr. R. P. Paranjpye as Hony. Librarian and Prof. D. D. Kapadia as Hony Joint Secretary, on the eve of their retirement from those offices :—

" That the Managing Committee have very great pleasure to place on record their high appreciation of the valuable services rendered to the Society by Mr. R. P. Paranjpye and by Prof. D. D. Kapadia, the former as Librarian from the beginning of the Society in 1907 and the latter as Joint Secretary from July 1910 to January 1922."

18, PYCROFT'S ROAD,
TRIPLICANE, MADRAS,
20-6-22.

P. V. SESHU AIYAR,
*Hony. Joint Secretary.*

# SRINIVASA RAMANUJAN:

OBITUARY NOTICE BY PROF. G. H. HARDY.*

## I

Srinivasa Ramanujan, who died at Kumbakonam † (sic !) on April 26th, 1920, had been a member of the Society since 1917. He was not a man who talked much about himself, and until recently I knew very little of his early life. Two notices, by P.V. Seshu Aiyar and R. Ramachandra Rao, two of the most devoted of Ramanujan's Indian friends, have been published recently in the *Journal of the Indian Mathematical Society*; and Sir Francis Spring has very kindly placed at my disposal an article which appeared in the *Madras Times* of April 5th, 1919. From these sources of information I can now supply a good many details with which I was previously unacquainted. Ramanujan (Srinivasa Iyengar Ramanuja Iyengar, to give him for once his proper name) was born on December 22nd, 1887, at Erode in Southern India. His father was an accountant (*gumasta*) to a cloth merchant at Kumbakonam, while his maternal grandfather had served as *amin* in the Munsiff's (or local Judge's) Court at Erode. He first went to school at five, and was transferred before he was seven to the Town High School at Kumbakonam, where he held a "free scholarship," and where his extraordinary powers appear to have been recognized immediately. "He used," so writes an old school-fellow to Mr. Seshu Aiyar, "to borrow Carr's *Synopsis of Pure Mathematics* from the College Library, and delight in verifying some of the formulæ given there......... He used to entertain his friends with his theorems and formulæ, even in those early days........... He had an extraordinary memory and could easily repeat the complete lists of Sanskrit roots (*atmanepadi* and *parasmaipadi*); he could give the values of $\sqrt{2}$, $\pi$, $e$,......to any number of decimal places......... In manners, he was simplicity itself...".

He passed his Matriculation examination at the Government College at Kumbakonam in 1904; and secured the "Junior Subramaniam Scholarship." Owing to weakness in English, he failed in his next examination and lost his scholarship; and left Kumbakonam, first for Vizagapatam and then for Madras. Here he presented himself for the "First Examination in Arts" in December 1906, but failed and never tried again. For the next few years he continued his independent work in Mathematics, "jotting down his results in two good-sized note books." I have one of these note-books in my possession still. In 1909, he married, and it became

* This appeared in the Proceedings of the London Mathematical Society and is quoted here with the permission of Prof. G. H. Hardy.

† This is not correct. He died at Chetput near Madras.

necessary for him to find some permanent employment. I quote Mr. Seshu Aiyar

"To this end, he went to Tirukoilur, a small sub-division town in South Arcot District, to see Mr. V. Ramaswami Aiyar, the Founder of the Indian Mathematical Society; but Mr. Aiyar, seeing his wonderful gifts, persuaded him to go to Madras. It was then after some four years' interval that Mr. Ramanujan met me at Madras, with his two well-sized note-books referred to above. I sent Ramanujan with a note of recommendation to that true lover of Mathematics, Dewan Bahadur R. Ramachandra Rao, who was then District Collector at Nellore, a small town some eighty miles north of Madras. Mr. Rao sent him back to me saying it was cruel to make an intellectual giant like Ramanujan rot at a mofussil station like Nellore, and recommended his stay at Madras, generously undertaking to pay Mr. Ramanujan's expenses for a time. This was in December 1910. After a while, other attempts to obtain for him a scholarship having failed, and Ramanujan himself being unwilling to be a burden on anybody for any length of time, he decided to take up a small appointment under the Madras Port Trust in 1911.

But he never slackened his work at Mathematics. His earliest contribution to the *Journal of the Indian Mathematical Society* was in the form of questions communicated by me in Vol. III (1911). His first long article on "Some properties of Bernoulli's Numbers" was published in the December number of the same volume. Mr. Ramanujan's methods were so terse and novel and his presentation was so lacking in clearness and precision, that the ordinary reader, unaccustomed to such intellectual gymnastics, could hardly follow him. This particular article was returned more than once by the Editor before it took a form suitable for publication. It was during this period that he came to me one day with some theorems on Prime Numbers, and when I referred him to Hardy's Tract on *Orders of Infinity*, he observed that Hardy had said on p. 36 of his Tract 'the exact order of $\rho(x)$ [defined by the equation

$$\rho(x) = \pi(x) - \int_2^x \frac{dt}{\log t}$$

where $\pi(x)$ denotes the number of primes less than $x$], has not yet been determined, and that he himself had discovered a result which gave the order of $\rho(x)$. On this, I suggested that he might communicate his result to Mr. Hardy, together with some more of his results."

This passage brings me to the beginning of my own acquaintance with Ramanujan. But before I say anything about the letters which I received from him, and which resulted ultimately in his journey to England,

I must add a little more about his Indian career. Dr. G. T. Walker, F.R.S., Head of the Meteorological Department, and formerly Fellow and Mathematical Lecturer of Trinity College, Cambridge, visited Madras for some official purpose some time in 1912; and Sir Francis Spring, K.C.I.E., the Chairman of the Madras Port Authority, called his attention to Ramanujan's work. Dr. Walker was too good a Mathematician not to recognize its quality, little as it had in common with his own. He brought Ramanujan's case to the notice of the Government and the University of Madras. A research studentship, "Rs. 75 *per mensem* for a period of two years," was awarded him; and he became and remained for the rest of his life, a professional Mathematician.

## II.

Ramanujan wrote to me first on January 16th, 1913, and at fairly regular intervals until he sailed for England in 1914. I do not believe that his letters were entirely his own. His knowledge of English, at that stage of his life, could scarcely have been sufficient, and there is an occasional phrase which is hardly characteristic. Indeed I seem to remember his telling that his friends had given him some assistance. However, it was the mathematics that mattered, and that was very emphatically his.

Madras, 16*th January* 1913.

"Dear Sir,

I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £ 20 per annum. I am now about 23 years of age. I have had no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course which is followed in a University course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I got are termed by the local mathematicians as "startling."

Just as in elementary mathematics you give a meaning to $a^n$ when $n$ is negative and fractional to conform to the law which holds when $n$ is a positive integer, similarly the whole of my investigations proceed on giving a meaning to the Eulerian Second Integral for all values of $n$. My friends who have gone through the regular course of University education tell me that $\int_0^\infty x^{n-1} e^{-x} dx = \Gamma(n)$ is true only when $n$ is positive. They say that this integral relation is not true when $n$ is negative. Supposing this is true only for positive values of $n$, and also supposing the definition

$n\,\Gamma(n) = \Gamma(n+1)$ to be universally true, I have given meanings to these integrals and under the conditions I state the integral is true for all values of $n$ negative and fractional. My whole investigations are based upon this and I have been developing this to a remarkable extent so much so that the local mathematicians are not able to understand me in my higher flights.

Very recently I came across a tract published by you, styled *Orders of Infinity*, in page 36 of which I find a statement that no definite expression has been as yet found for the number of prime numbers less than any given number. I have found an expression which very nearly approximates to the real result, the error being negligible. I would request you to go through the enclosed papers. Being poor, if you are convinced that there is anything of value, I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get; but I have indicated the lines on which I proceed. Being inexperienced, I would very highly value any advice you give me. Requesting to be excused for the trouble I give you,

I remain, Dear Sir,
Yours truly,
S. Ramanujan.

P. S. My address is S. Ramanujan, Clerk, Accounts Department, Port Trust, Madras, India."

I quote now from the "papers enclosed" and from later letters :—

"In page 36, it is stated that 'the number of prime numbers less than $x$

$$= \int_2^x \frac{dt}{\log t} + \rho(x),$$

where the precise order of $\rho(x)$ has not been determined.........

I have observed that $\rho(e^{\pi x})$ is of such a nature that its value is very small when $x$ lies between 0 and 3 (its value is less than a few hundreds when $x = 3$) and rapidly increases when $x$ is greater than 3 .........

The difference between the number of prime numbers of the form $4n-1$ and which are less than $x$ and those of the form $4n+1$ less than $x$ is infinite when $x$ becomes infinite.

... ... ... ... ...

The following are a few examples from my theorems :—

(1) The nos. of the form $2^p\,3^q$ less than $n$

$$= \frac{1}{2}\,\frac{\log(2n)\log(3n)}{\log(2)\log(3)},$$ where $p$ and $q$ may have any positive integral values including zero.

(2) Let us take all nos. containing an odd no. of dissimilar prime divisors, *viz.*

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 30, 31, 37, 41, 42, 43, 47, &c.

(*a*) The no. of such nos. less than $n = \frac{3n}{\pi^2}$.

(*b*) $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots + \frac{1}{30^2} + \frac{1}{31^2} + \ldots = \frac{9}{2\pi^2}$.

(*c*) $\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots = \frac{15}{2\pi^4}$.

(3) Let us take the no. of divisors of natural nos., *viz.*, 1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, &c. (1 having 1 divisor, 2 having 2, 3 having 2, 4 having 3, 5 having 2, &c.).

The sum of such nos. to $n$ terms
$= n\,(2\gamma - 1 + \log n) + \frac{1}{2}$ of the no. of divisors of $n$,
where $\gamma = \cdot 5772156649 \ldots$ the Eulerian constant.

(4) 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, &c., are nos. which are either themselves sqq., or which can be expressed as the sum of two sqq.

The no. of such nos. greater than A and less than B

$$= K \int_{A}^{B} \frac{dx}{\sqrt{\log x}} + \theta\,(x)^{*},$$

where $K = \cdot 764\ldots$ and $\theta\,(x)$ is very small when compared with the previous integral. $K$ and $\theta\,(x)$ have been exactly found though complicated.........".

Ramunujan's theory of primes was vitiated by his ignorance of the theory of functions of a complex variable. It was (so to say) what the theory might be if the Zeta-function had no complex zeros. His methods of proof depended upon a wholesale use of divergent series. He disregarded entirely all the difficulties which are involved in the interchange of double limit operations; he did not distinguish, for example, between the sum of a series $\Sigma\, a_n$ and the value of the Abelian limit

$$\lim_{x \to 1} \sum a_n x^n$$

or that of any other limit which might be used for similar purposes by a modern analyst. There are regions of mathematics in which the precepts of modern rigour may be disregarded with comparative safety, but the

* This should pesumably be $\theta$ (B).

Analytic Theory of Numbers is not one of them, and Ramanujan's Indian work on primes, and on all the allied problems of the theory, was definitely wrong. That his proofs should have been invalid was only to be expected. But the mistakes went deeper than that, and many of the actual results were false. He had obtained the dominant terms of the classical formulæ, although by invalid methods, but none of them are such close approximations as he supposed.

This may be said to have been Ramanujan's one great failure. And yet I am not sure that, in some ways, his failure was not more wonderful than any of his triumphs. Consider, for example, problem (4). The dominant term, which Ramanujan gives correctly, was first obtained by Landau in 1908. The correct order of the error term is still unknown. Ramanujan had none of Landau's weapons at his command; he had never seen a French or German book; his knowledge even of English was insufficient to enable him to qualify for a degree. It is sufficiently marvellous that he should have even dreamt of problems such as these, problems which it has taken the finest mathematicians in Europe a hundred years to solve, and of which the solution is incomplete to the present day.

"IV. *Theorems on Integrals.* The following are a few examples:—

(1) $$\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \cdots\cdots\, dx$$

$$= \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)} \cdot \frac{\Gamma(b+1)}{\Gamma(b+\frac{1}{2})} \cdot \frac{\Gamma(b-a+\frac{1}{2})}{\Gamma(b-a+1)} \cdots\cdots$$

(3) If $$\int_0^\infty \frac{\cos nx\, dx}{e^{2\pi\sqrt{x}}-1} = \phi(n),$$

then $$\int_0^\infty \frac{\sin nx}{e^{2\pi\sqrt{x}}-1}\, dx = \phi(n) - \frac{1}{2n} + \phi\left(\frac{\pi^2}{n}\right)\sqrt{\frac{2\pi^3}{n^3}}.$$

$\phi(n)$ is a complicated function. The following are certain special values:—

$$\phi(0) = \frac{1}{12};\ \phi\left(\frac{\pi}{2}\right) = \frac{1}{4\pi};\ \phi(\pi) = \frac{2-\sqrt{2}}{8};\ \phi(2\pi) = \frac{1}{16}.$$

$$\phi\left(\frac{2\pi}{5}\right) = \frac{8-8\sqrt{5}}{16};\ \phi\left(\frac{\pi}{5}\right) = \frac{6+\sqrt{5}}{4} - \frac{5\sqrt{10}}{8};\ \phi(\infty)=0.$$

$$\phi\left(\frac{2\pi}{3}\right) = \frac{1}{3} - \sqrt{3}\left(\frac{3}{16} - \frac{1}{8\pi}\right).$$

(4) $$\int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\ldots \&c.}$$

$$= \frac{\pi}{2(1+r+r^3+r^6+r^{10}+\ldots \&c.)}$$

where 1, 3, 6, 10...are the sums of natural nos.

(5) $$\int_0^\infty \frac{\sin 2nx\,dx}{x(\cosh \pi x + \cos \pi x)} =$$

$$\frac{\pi}{4} - 2\left\{\frac{e^{-n}\cos n}{\cosh \frac{\pi}{2}} - \frac{e^{-3n}\cos 3n}{3\cosh \frac{3\pi}{2}} \quad \&c.\right\}$$

V. *Theorems on Summation of Series ; e.g.,**

(1) $$\frac{1}{1^3}\cdot\frac{1}{2} + \frac{1}{2^3}\cdot\frac{1}{2^2} + \frac{1}{3^3}\cdot\frac{1}{2^3} + \frac{1}{4^3}\cdot\frac{1}{2^4} + \;\&c.$$

$$= \frac{1}{6}(\log 2)^3 - \frac{\pi^2}{12}\log 2 + \left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\ldots\;\&c.\right)$$

(2) $$1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1\cdot5}{4\cdot6}\right)^4 + 25\left(\frac{1\cdot5\cdot9}{4\cdot8\cdot12}\right)^4 + \&c.$$

$$= \frac{2\sqrt{2}}{\sqrt{\pi}\cdot\left\{e\;\left(\frac{3}{4}\right)^2\right\}^2}.$$

(3) $$1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1\cdot3}{2\cdot4}\right)^3 - \&c. = \frac{2}{\pi}.$$

(4) $$\frac{1^{13}}{e^{2\pi}-1} + \frac{2^{13}}{e^{4\pi}-1} + \frac{3^{13}}{e^{6\pi}-1} + \&c. = \frac{1}{24}.$$

(5) $$\frac{\coth \pi}{1^7} + \frac{\coth 2\pi}{2^7} + \frac{\coth 3\pi}{3^7} + \&c. = \frac{19\pi^7}{56700}.$$

$$\frac{1}{1^5\cosh\frac{\pi}{2}} - \frac{1}{3^5\cosh\frac{3\pi}{2}} + \frac{1}{5^5\cosh\frac{5\pi}{2}} - \&c. = \frac{\pi^5}{768}.$$

... ... ... ...

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* There is always more in one of Ramanujam's formulæ than meets the eye, as anyone who sets to work to verify those which look the easiest will soon discover. In some the interest lies very deep, in others comparatively near the surface ; but there is not one which is not curious and entertaining.

VI. *Theorems on Transformations of Series and Integrals*; *e.g.*

(1) $$\pi\left(\frac{1}{2}-\frac{1}{\sqrt{1}+\sqrt{3}}-\frac{1}{\sqrt{3}+\sqrt{5}}-\frac{1}{\sqrt{5}+\sqrt{7}}+\&c.\right)$$

$$=\frac{1}{1\sqrt{1}}-\frac{1}{3\sqrt{3}}+\frac{1}{5\sqrt{5}}-\&c.$$

. . . . . . . . . . .

(3) $$1-\frac{x^2\,3\,!}{(1\,!\,2\,!)^3}+\frac{x^4\,6\,!}{(2\,!\,4\,!)^3}-\frac{x^6\,9\,!}{(3\,!\,6\,!)^3}+\&c.$$

$$=\left\{1+\frac{x}{(1\,!)^3}+\frac{x^2}{(2\,!)^3}+\&c.\right\}\left\{1-\frac{x}{(1\,!)^3}+\frac{x^2}{(2\,!)^3}\ldots\&c.\right\}$$

(6) If $\alpha\beta=\pi^2$, then $$\frac{1}{\sqrt[4]{\alpha}}\left\{1+4\alpha\int_0^\infty\frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,dx\right\}$$

$$=\frac{1}{\sqrt[4]{\beta}}\left\{1+4\beta\int_0^\infty\frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\,dx\right\}.$$

(7) $$n\left\{e^{-n^2}-\frac{e^{-\frac{n^2}{3}}}{3\sqrt{3}}+\frac{e^{-\frac{n^2}{5}}}{5\sqrt{5}}-\&c.\right\}$$

$$=\sqrt{\pi}\left(e^{-n\sqrt{\pi}}\sin n\sqrt{\pi}-e^{-n\sqrt{3\pi}}\sin n\sqrt{3\pi}+\ldots\&c.\right)$$

(8) If $n$ is any positive integer excluding zero,

$$\frac{1^{4n}}{(e^{\pi}-e^{-\pi})^2}+\frac{2^{4n}}{(e^{2\pi}-e^{-2\pi})^2}+\ldots\&c.$$

$$=\frac{n}{\pi}\left\{\frac{B_{4n}}{8n}+\frac{1^{4n-1}}{e^{2\pi}-1}+\frac{2^{4n-1}}{e^{4\pi}-1}-\ldots\&c.\right\}$$

where $B_2=\frac{1}{6}$, $B_4=\frac{1}{30}$, &c.

VII. *Theorems on approximate integration and summation of series*:

. . . . . . . . . . .

(2) $$1+\frac{x}{1\,!}+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots+\frac{x^x}{x\,!}\,\theta=\frac{e^x}{2},$$

where $\theta=\frac{1}{3}+\frac{4}{135\,(x+k)}$, where $k$ lies between $\frac{8}{45}$ and $\frac{2}{21}$.

(3) $$1+\left(\frac{x}{1\,!}\right)^5+\left(\frac{x^2}{2\,!}\right)^5+\left(\frac{x^3}{3\,!}\right)^5+\&c.$$

$$=\frac{\sqrt{5}}{4\pi^2}\cdot\frac{e^{5x}}{5x^2-x+\theta},$$

where $\theta$ vanishes when $x=\infty$.

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(4) $\frac{1^2}{e^x-1}+\frac{2^2}{e^{2x}-1}+\frac{3^2}{e^{3x}-1}+\frac{4^2}{e^{3x}-1}+\ldots$ &c.

$$=\frac{2}{x^3}\left(\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\text{ \&c.}\right)-\frac{1}{12x}-\frac{x}{1440}+\frac{x^3}{181440}$$

$$+\frac{x^5}{7257600}+\frac{x^7}{159667200}+\text{ \&c., when } x \text{ is small.}$$

NOTE.—$x$ may be given values from 0 to 2).

(5) $\frac{1}{1001}+\frac{1}{1002^2}+\frac{3}{1003^3}+\frac{4^2}{1004^4}+\frac{5^3}{1005^5}+$ &c.

$$=\frac{1}{1000}-10^{-440}\times 1{\cdot}0125 \text{ nearly.}$$

(6) $\int_0^a e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}-\frac{e^{-a^2}}{2a+}\;\frac{1}{a+}\;\frac{2}{2a+}\;\frac{3}{a+}\;\frac{4}{2a+}$ &c.

(7) The co-efficient of $x^n$ in $\frac{1}{1-2x+2x^4-2x^9+2x^{16}-\ldots}$

$$=\text{the nearest integer to } \frac{1}{4n}\left\{\cosh\left(\pi\sqrt{n}\right)-\frac{\sinh\left(\pi\sqrt{n}\right)}{\pi\sqrt{n}}\right\}.^{*}$$

VIII. *Theorems on continued fractions*: a few example are:—

(1) $\frac{4}{x+}\;\frac{1^2}{2x+}\;\frac{3^2}{2x+}\;\frac{5^2}{2x+}\;\frac{7^2}{2x+\text{ \&c.}}=\left\{\frac{\Gamma\left(\frac{x+1}{4}\right)}{\Gamma\left(\frac{x+3}{4}\right)}\right\}^2.$

... ... ...

(4) If $u=\frac{x}{1+}\;\frac{x^5}{1+}\;\frac{x^{10}}{1+}\;\frac{x^{15}}{1+}\;\frac{x^{20}}{1+\text{ \&c.}}$

and $v=\frac{\sqrt[5]{x}}{1+}\;\frac{x}{1+}\;\frac{x^2}{1+}\;\frac{x^3}{1+\text{ \&c.}}$

then $v^5=u\cdot\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}.$

(5) $\frac{1}{1+}\;\frac{e^{-2\pi}}{1+}\;\frac{e^{-4\pi}}{1+}\;\frac{e^{-6\pi}}{1+\text{ \&c.}}$

$$=\left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+1}{2}\right)e^{\frac{2\pi}{5}}$$

* This is quite untrue. But the formula is extremely interesting for a variety of reasons.

(6) $\frac{1}{1-}\frac{e^{-\pi}}{1+}\frac{e^{-2\pi}}{1-}\frac{e^{-3\pi}}{1+\&c.}$

$$= \left(\sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}\right) e^{\frac{\pi}{5}}$$

(7) $\frac{1}{1+}\frac{e^{-\pi\sqrt{n}}}{1+}\frac{e^{-2\pi\sqrt{n}}}{1+}\frac{e^{-3\pi\sqrt{n}}}{1+\&c.,}$

can be exactly found if $n$ be any positive rational quantity."

"27th February 1913.

"...... I have found a friend in you who views my labours sympathetically. This is already some encouragement to me to proceed ........... I find in many a place in your letter rigorous proofs are required and you ask me to communicate the methods of proof....I told him* that the sum of an infinite no. of terms of the series $1+2+3+4+\ldots = -\frac{1}{12}$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal .....What I tell you is this. Verify the results I give and if they agree with your results ... you should at least grant that there may be some truths in my fundamental basis......

To preserve my brains, I want food and this is now my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the University or from the Government........."

1. The no. of prime nos. less than $e^a$

$$= \int_0^\infty \frac{a^x \, dx}{x\, S_{x+1}\, \Gamma(x+1)}$$

where $S_{x+1} = \frac{1}{1^{x+1}} + \frac{1}{2^{x+1}} + \frac{1}{3^{x+1}} + \ldots\ldots\ldots\ldots$

2. The no. of prime nos. less than $n$

$$= \frac{2}{\pi}\left\{\frac{2}{B_2}\left(\frac{\log n}{2\pi}\right) + \frac{4}{3B_4}\left(\frac{\log n}{2\pi}\right)^3 + \frac{6}{5B_6}\left(\frac{\log n}{2\pi}\right)^5 + \&c.\right\}$$

where $B_2 = \frac{1}{6}$; $B_4 = \frac{1}{30}$. &c., are the Bernoullian nos.

... ... ... ...

For practical calculations

$$\int_\mu^n \frac{dx}{\log x} = n\left(\frac{1}{\log n} + \frac{1!}{(\log n)^2} + \ldots\ldots + \frac{\overline{k-1}!}{(\log n)^k}\,\theta\right)$$

* Referring to a previous correspondence.

$$\text{where } \theta = \tfrac{2}{3} - \delta + \frac{1}{\log n}\left\{\frac{4}{135} - \frac{\delta^2(1-\delta)}{3}\right\}$$

$$+ \frac{1}{(\log n)^2}\left\{\frac{8}{2835} + \frac{2\delta(1-\delta)}{135} - \frac{\delta(1-\delta^2)(2-3\delta^2)}{45}\right\}$$

$$+ \&c.$$

where $\delta = k - \log n$ ......

The order of $\theta(x)$ which you asked in your letter is

$$\sqrt{\left(\frac{x}{\log x}\right)}.$$

... ... ... ...

(1) If $F(x) = \frac{1}{1+}\;\frac{x}{1+}\;\frac{x^2}{1+}\;\frac{x^3}{1+}\;\frac{x^4}{1+}\;\frac{x^5}{1+, \&c.}$

then
$$\left\{\frac{\sqrt{5}+1}{2} + e^{-\frac{2\alpha}{5}} F(e^{-2\alpha})\right\} \times$$

$$\left\{\frac{\sqrt{5}+1}{2} + e^{-\frac{2\beta}{5}} F(e^{-2\beta})\right\} = \frac{5+\sqrt{5}}{2},$$

with the conditions $\alpha\beta = \pi^2$......... *e. g.*

$$\frac{1}{1+}\;\frac{e^{-2\pi\sqrt{5}}}{1+}\;\frac{e^{-4\pi\sqrt{5}}}{1+\&c.}$$

$$= e^{\frac{2\pi}{\sqrt{5}}}\left\{\frac{\sqrt{5}}{1+\sqrt[5]{5^{\frac{3}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}}-1}} - \frac{\sqrt{5}+1}{2}\right\}$$

The above theorem is a particular case of a theorem on the c. f.

$$\frac{1}{1+}\;\frac{ax}{1+}\;\frac{ax^2}{1+}\;\frac{ax^3}{1+}\;\frac{ax^4}{1+}\;\frac{ax^5}{1+\&c.}$$

which is a particular case of the c. f.

$$\frac{1}{1+}\;\frac{ax}{1+bx+}\;\frac{ax^2}{1+bx^2+}\;\frac{ax^3}{1+bx^3+\&c.}$$

which is a particular case of a general theorem on c.f.

(2) (i) $4\int_0^\infty \frac{xe^{-x\sqrt{5}}}{\cosh x}\,dx = \frac{1}{1+}\;\frac{1^2}{1+}\;\frac{1^2}{1+}\;\frac{2^2}{1+}\;\frac{2^2}{1+}\;\frac{3^2}{1+}\;\frac{3^2}{1+\&c.}$

(ii) $4\int_0^\infty \frac{x^2e^{-x\sqrt{3}}}{\cosh x}\,dx = \frac{1}{1+}\;\frac{1^3}{1+}\;\frac{1^3}{3+}\;\frac{2^3}{1+}\;\frac{2^3}{5+}\;\frac{3^3}{1+}\;\frac{3^3}{7+\&c.}$

(3) $$1-5\left(\tfrac{1}{2}\right)^5+9\left(\frac{1\cdot3}{2\cdot4}\right)^5-13\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^5+\&c.$$
$$=\frac{2}{\{\Gamma(\frac{3}{4})\}^4}.$$

... ... ... ... ...

(6) If $$v=\frac{x}{1+}\;\frac{x^3+x^6}{1+}\;\frac{x^6+x^{12}}{1+}\;\frac{x^9+x^{18}}{1+\&c.}$$

then (i) $$x\left(1+\frac{1}{v}\right)=\frac{1+x+x^3+x^6+x^{10}+\&c.}{1+x^9+x^{27}+x^{54}+x^{90}+\&c.}$$

(ii) $$x^3\left(1+\frac{1}{v^3}\right)=\left(\frac{1+x+x^3+x^6+x^{10}+\&c.}{1+x^3+x^9+x^{18}+x^{30}+\&c.}\right)^4$$

(7) If $n$ is any odd integer,

$$\frac{1}{\cosh\frac{\pi}{2n}+\cos\frac{\pi}{2n}}-\frac{1}{3\left(\cosh\frac{3\pi}{2n}+\cos\frac{3\pi}{2n}\right)}$$
$$+\frac{1}{5\left(\cosh\frac{5\pi}{2n}+\cos\frac{5\pi}{2n}\right)}-\ldots=\frac{\pi}{8}.$$

(10) If $F(\alpha,\beta,\gamma,\delta,t)=$

$$1+\frac{\alpha}{1!}\cdot\frac{\beta}{\delta}\cdot\frac{\gamma}{t}+\frac{\alpha(\alpha+1)}{2!}\,\frac{\beta(\beta+1)}{\delta(\delta+1)}\times\frac{\gamma(\gamma+1)}{t(t+1)}+\&c.$$

then $$F(\alpha,\beta,\gamma,\delta,t)=\frac{\Gamma(\delta)\,\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\,\Gamma(\delta-\beta)}\times$$
$$F(\alpha,\beta,t-\gamma,\alpha+\beta-\delta+1,t)$$
$$+\frac{\Gamma(\delta)\,\Gamma(t)\,\Gamma(\alpha+\beta-\delta)\,\Gamma(\delta+t-\alpha-\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(t-\gamma)\,\Gamma(\delta+t-\alpha-\beta)}\times$$
$$F(\delta-\alpha,\delta-\beta,\delta+t-\alpha-\beta-\gamma,$$
$$\delta-\alpha-\beta+1,\delta+t-\alpha-\beta).$$

(13) $$\frac{a}{1+n+}\;\frac{a^2}{3+n+}\;\frac{(2a)^2}{5+n+}\;\frac{(3a)^2}{7+n+\ldots}$$
$$=2a\int_0^1 z^{\frac{n}{\sqrt{1+a^2}}}\frac{dz}{\{\sqrt{(1+a^2)}+1\}+z^2\{\sqrt{1+a^2}-1\}}$$

(14) If $$F(\alpha,\beta)=\alpha+\frac{(1+\beta)^2+k}{2\alpha+}\;\frac{(3+\beta)^2+k}{2\alpha+}\;\frac{(5+\beta)^2+k}{2\alpha+\ldots},$$

then $$F(\alpha,\beta)=F(\beta,\alpha).$$

(15) If $F(\alpha, \beta) = \frac{\alpha}{n+} \frac{\beta^2}{n+} \frac{(2\alpha)^2}{n+} \frac{(3\beta)^2}{n+\ldots}$

then $F(\alpha, \beta) + F(\beta, \alpha) = 2F\left\{\frac{1}{2}(\alpha+\beta), \sqrt{\alpha\beta}\right\}$.

... ... ...

(17) If $F(k) = 1 + \left(\frac{1}{2}\right)^2 k + \left(\frac{1\cdot3}{2\cdot4}\right)^2 k^2 + \ldots\ldots$

and $F(1-k) = \sqrt{210}\, F(k)$,

then $k = (\sqrt{2}-1)^4 (2-\sqrt{3})^2 (\sqrt{7}-\sqrt{6})^4 (8-3\sqrt{7})^2 (\sqrt{10}-3)^4 \times$

$(4-\sqrt{15})^4 (\sqrt{15}-\sqrt{14})^2 (6-\sqrt{35})^2$.

... ... ...

(20) If $F(\alpha) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{\{1-(1-\alpha)\sin^2\phi\}}} \div$

$\int_0^{\pi/2} \frac{d\phi}{\sqrt{\{1-\alpha\sin^2\phi\}}}$,

and $F(\alpha) = 3F(\beta) = 5F(\gamma) = 15F(\delta)$,

then (i) $[(\alpha\delta)^{\frac{1}{8}} + \{(1-\alpha)(1-\delta)\}^{\frac{1}{8}}][(\beta\gamma)^{\frac{1}{8}}$

$+ \{(1-\beta)(1-\gamma)\}^{\frac{1}{8}}] = 1$;

... ... ...

(v) $[\alpha\beta\gamma\delta]^{\frac{1}{8}} + \{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{8}}$

$+ \{16\,\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{24}} = 1$.

... ... ...

(21) If $F(\alpha) = 3F(\beta) = 13F(\gamma) = 39F(\delta)$,

or $F(\alpha) = 5F(\beta) = 11F(\gamma) = 55F(\delta)$,

or $F(\alpha) = 7F(\beta) = 9F(\gamma) = 63F(\delta)$,

then $\frac{\{(1-\alpha)(1-\delta)\}^{\frac{1}{8}} - (\alpha\delta)^{\frac{1}{8}}}{\{(1-\beta)(1-\gamma)\}^{\frac{1}{8}} - (\beta\gamma)^{\frac{1}{8}}} = \frac{1+\{(1-\alpha)(1-\delta)\}^{\frac{1}{8}} + (\alpha\delta)^{\frac{1}{4}}}{1+\{(1-\beta)(1-\gamma)\}^{\frac{1}{8}} + (\beta\gamma)^{\frac{1}{4}}}$.

... ... ...

$$(23)\quad \left(1+e^{-\pi\sqrt{1353}}\right)\left(1+e^{-3\pi\sqrt{1353}}\right)\left(1+e^{-5\pi\sqrt{1353}}\right)\ldots\ldots$$

$$= \sqrt[4]{2}\, e^{-\frac{1}{24}\pi\sqrt{1353}} \times$$

$$\sqrt{\left\{\sqrt{\left(\frac{569+99\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{561+99\sqrt{33}}{8}\right)}\right\}} \times$$

$$\sqrt{\left\{\sqrt{\left(\frac{25+3\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{17+3\sqrt{33}}{8}\right)}\right\}} \times$$

$$\sqrt[4]{\left(\frac{\sqrt{123}+11}{\sqrt{2}}\right)} \times \sqrt[8]{(10+3\sqrt{11})}$$

$$\times \sqrt[8]{(26+15\sqrt{3})} + \sqrt[12]{\left(\frac{6817+321\sqrt{451}}{\sqrt{2}}\right)}.$$

... ... ... ...

"17 April 1913.

"...... I am a little pained to see what you have written......* I am not in the least apprehensive of my method being utilized by others. On the contrary, my method has been in my possession for the last eight years and I have not found anyone to appreciate the method. As I wrote in my last letter, I have found a sympathetic friend in you and I am willing to place unreservedly in your hands what little I have. It was on account of the novelty of the method I have used that I am a little diffident even now to communicate my own way of arriving at the expressions I have already given .............. I am glad to inform you that the local University has been pleased to grant me a scholarship of £60 per annum for two years and this was at the instance of Dr. Walker, F. R. S., Head of the Meteorological Department in India, to whom my thanks are due...... I request you to convey my thanks also to Mr. Littlewood, Dr. Barnes, Mr. Berry and others who take an interest in me..........."

## III

It is unnecessary to repeat the story of how Ramanujan was brought to England. There were serious difficulties; and the credit for overcoming them is primarily due to Prof. E. H. Neville, in whose company Ramanujan arrived in April 1914. He had a scholarship from Madras of £250, of which £50 was allotted to the support of his family in India, and an exhibition of £60 from Trinity. For a man of his almost ludicrously simple tastes, this was an ample income, and he was able to save

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* Ramanujan might very reasonably have been reluctant to give away his secrets to an English mathematician, and I had tried to reassure him on this point as well as I could.

a good deal of money which was badly wanted later. He had no duties and could do as he pleased ; he wished indeed to qualify for a Cambridge Degree as a research student, but this was a formality. He was now, for the first time in his life, in a really comfortable position and could devote himself to his researches without anxiety.

There was one great puzzle. What was to be done in the way of teaching him modern mathematics ? The limitations of his knowledge were as startling as its profundity. Here was a man who could work out modular equations, and theorems of complex multiplication, to orders unheard of, whose mastery of continued fractions was, on the formal side at any rate, beyond that of any mathematician in the world, who had found for himself the functional equations of the Zeta-functions, and the dominant terms of many of the most famous problems in the analytic theory of numbers ; and he had never heard of a doubly periodic function or of Cauchy's theorem, and had indeed but the vaguest idea of what a function of a complex variable was. His ideas as to what constituted a mathematical proof were of the most shadowy description. All his results, new or old, right or wrong, had been arrived at by a process of mingled argument, intuition, and induction, of which he was entirely unable to give any coherent account.

It was impossible to ask such a man to submit to systematic instruction, to try to learn mathematics from the beginning once more. I was afraid that, if I insisted unduly on matters which Ramanujan found irksome, I might destroy his confidence or break the spell of his inspiration. On the other hand, there were things of which it was impossible that he should remain in ignorance. Some of his results were wrong and in particular those which concerned the distribution of primes, to which he attached the greatest importance. It was impossible to allow him to go through life supposing that all the zeros of the Zeta-function were real. So I had to try to teach him, and in a measure, I succeeded, though obviously I learnt from him much more than he learnt from me. In a few years' time, he had a tolerable knowledge of the theory of functions and the analytic theory of numbers. He was never a mathematician of the modern school, and it was hardly desirable that he should become one ; but he knew when he had proved a theorem and when he had not. And his flow of original ideas showed no symptom of abatement.

I should add a word here about Ramanujan's interests outside mathematics. Like his mathematics, they showed the strangest contrasts. He had very little interest, I should say, in literature as such, or in art though he could tell good literature from bad. On the other hand he was

a keen philosopher, of what appeared, to followers of the modern Cambridge schools a rather nebulous kind, and an ardent politician, of a pacifist and ultra-radical type. He adhered with a severity most unusual in Indians resident in England, to the religious observances of his caste; but his religion was a matter of observance and not of intellectual conviction, and I remember well his telling me (much to my surprise) that all religions seemed to him more or less equally true. Alike in literature, philosophy, and mathematics, he had a passion for what was unexpected, strange, and odd, he had quite a small library of books by circle-squarers and other cranks.

It was in the spring of 1917 that Ramanujan first appeared to be unwell. He went into the Nursing Home at Cambridge in the early summer, and was never out of bed for any length of time again. He was in Sanatoria at Wells, at Matlock, and in London, and it was not until the autumn of 1918 that he showed any decided symptom of improvement. He had then resumed active work, stimulated perhaps by his election to the Royal Society, and some of his most beautiful theorems were discovered about this time. His election to a Trinity Fellowship was a further encouragement; and each of those famous societies may well congratulate themselves that they recognized his claims before it was too late. Early in 1919, he had recovered, it seemed, sufficiently for the voyage home to India, and the best medical opinion held out hopes of a permanent restoration. I was rather alarmed by not hearing from him for a considerable time; but a letter reached me in February 1920, from which it appeared that he was still active in research.

"University of Madras, 12*th February* 1920.

"I am extremely sorry for not writing you a single letter up to now.

... ... ...

I discovered very interesting functions recently which I call "Mock" $\vartheta$-functions. Unlike the "False" $\vartheta$-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary $\vartheta$-functions. I am sending you with this letter some examples.........

*Mock $\vartheta$-functions.*

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \dots\dots$$

$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \dots$$

*Mock* $\mathfrak{D}$*-functions (of 5th order).*

$$f(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \frac{q^9}{(1+q)(1+q^2)(1+q^3)} + \dots$$

... ... ...

*Mock* $\mathfrak{D}$*-functions (of 7th order)*

$$\text{(i)}\quad 1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)} + \dots$$

... ... ... ...

He said little about his health, and what he said was not particularly discouraging; and I was quite unprepared for the news of his death.

## IV.

Ramanujan published the following papers in Europe:—

(1) "Some definite integrals," *Messenger of Mathematics*, Vol. 44 (1914), pp. 10—18.

(2) "Some definite integrals connected with Gauss's sums," *ibid.* pp. 75—85.

(3) "Modular equations and approximations to $\pi$," *Quarterly Journal of Mathematics*, Vol. 45 (1914), pp. 350-372.

(4) "New expressions for Riemann's functions $\zeta(s)$ and $\Xi(t)$" *ibid*, Vol. 46 (1915), pp. 253—261.

(5) "On certain infinite series," *Messenger of Mathematics*, Vol. 45 (1915), pp. 11—15.

(6) "Summation of a certain series", *ibid*, pp. 157—160.

(7) "Highly Composite Numbers," *Proc. London Math. Soc.*, Ser. 2, Vol. 14 (1915) pp. 347—409.

(8) "Some formulæ in the analytical theory of numbers," *Messenger of Mathematics*, Vol. 45 (1916) pp. 81—84.

(9) "On certain arithmetical functions," *Trans. Cambridge Phil. Soc*, Vol. 22 (1916), No. 9, pp. 159—184.

(10) "Some Series for Euler's Constant," *Messenger of Mathematics*, Vol. 46 (1916), pp. 73—80.

(11) "On the expression of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$" *Proc. Cambridge Phil. Soc.*, Vol. 19 (1917), pp. 11—21.

*(12) "Une formule asymptotique pour le nombre des partitions de *n*," *Comptes Rendus*, 2nd Jan. 1917.

*(13) "Asymptotic formulæ concerning the distribution of integers of various types," *Proc. London Math. Soc.* Ser. 2, Vol. 16 (1917), pp. 112—132.

*(14) "The normal number of prime factors of a number *n*". *Quarterly Journal of Mathematics.* Vol. 48 (1917). pp. 76—92.

*(15) "Asymptotic formulæ in Combinatory Analysis," *Proc. London Math. Soc.*, Ser. 2, Vol. 17 (1918), pp. 75—115.

*(16) "On the co-efficients in the expansioning of certain modular functions." *Proc. Royal Soc.* (A), Vol. 95 (1918), pp. 144—155.

(17) "On certain trigonometrical sums and their applications in the theory of numbers," *Trans. Camb. Phil. Soc.*, Vol. 22 (1918), pp. 259—276.

(18) "Some properties of $p(n)$, the number of partition of *n*." *Proc. Camb. Phil. Soc.*, Vol. 19 (1919), pp. 207, 210.

(19) "Proof of certain identities in Combinatory Analysis." *ibid*, pp. 214—216.

(20) "A class of definite integrals," *Quarterly Journal of Mathematics*, Vol. 48 (1920), pp. 294—309.

(21) "Congruence properties of partitions," *Math. Zeitschrift*, Vol. 9 (1921), pp. 147—153.

Of these, those marked with an asterisk were written in collaboration with me, and (21) is a posthumous extract from a much larger unpublished manuscript in my possession †. He also published a number of short notes in the *Records of Proceedings* at our meetings, and in the *Journal of the Indian Mathematical Society*. The complete list of these is as follows.—

### Records of Proceedings at Meetings.

*(22) "Proof that almost all numbers *n* are composed of about log log *n* prime factors." 14th Dec. 1916.

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† All of Ramanujan's manuscripts passed through my hands, and I edited them very carefully for publication. The earlier ones I re-wrote completely. I had no share of any kind in the results, except of course when I was a collaborator, or when explicit acknowledgment is made. Ramanujan was almost absurdly scrupulous in his desire to acknowledge the slightest help.

*(23) "Asymptotic formulæ in Combinatory Analysis" 1st March, 1917.

(24) "Some Definite Integrals", 17th Jan. 1918.

(25) "Congruence properties of partitions." 13th March 1919.

(26) "Algebraic relations between certain infinite products." 13th March 1919.

## Journal of the Indian Mathematical Society.

### A. Articles and Notes.

(27) "Some properties of Bernoulli's numbers." Vol. 3 (1911), pp. 219—235.

(28) "On Q. 330 of Prof. Sanjana." Vol. 4 (1912) pp. 59—61.

(29) "A set of equations." Vol. 4 (1912), pp. 94—96.

(30) "Irregular numbers." Vol. 5 (1913), pp. 105—107.

(31) "Squaring the circle" Vol. 5 (1913), pp. 132—133.

(32) "On the Integral $\int_0^x \text{arc tan}\, t \,.\, \frac{dt}{t}$". Vol. 7 (1915) pp. 93—96.

(33) "On the divisors of a number." Vol. 7 (1915), pp. 131—134.

(34) "The sum of the square roots of the first $n$ natural numbers." Vol. 7 (1915), pp. 173—175.

(35) "On the product $\Pi\left(1+\frac{x^2}{(a+nd)^2}\right)$" Vol. 7 (1915), pp. 209—212.

(36) "Some definite integrals." Vol. 11 (1919), pp. 81—88.

(37) "A proof of Bertrand's postulate." Vol. 11 (1919), pp. 181—183.

(38) "Communicated by S. Narayana Iyer." Vol. 3 (1911) p. 60.

### B. Questions proposed and solved.

Nos. 260, 261, 283, 289, 294, 295, 298, 308, 353, 358, 386, 427, 441, 464, 489, 507, 541, 546, 571, 605, 606, 629, 642, 666, 682, 700, 723, 724, 739 740, 753, 768, 769, 783, 785.

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C. Questions proposed but not solved as yet.

Nos. 284, 327, 359, 387, 441, 463, 469, 524, 525, 526, 584, 661, 662, 681, 699, 722, 738, 754, 770, 784, 1049, 1070, 1076.

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Finally, I may mention the following writings by other authors, concerned with Ramanujan's work :—

'Proof of a formula of Mr. Ramunujan," by G. H. Hardy. (*Messenger of Mathematics*, Vol. 44, 1915, pp. 18—21).

"Mr. Ramanujan's mathematical work in England," by G.H. Hardy. (Report to the University of Madras, 1916, privately printed).

"On Mr. Ramanujan's Empirical expansions of modular functions," by L. J. Mordell. (*Proc. Camb. Phil. Soc.* Vol. 19, 1917, pp. 117—124.

"Life Sketch of Ramanujan" (Editorial in the *Journal of the Indian Math. Soc.*, Vol. 11, 1919, p. 122)

"Note on the parity of the number which enumerates the partitions of a number," by P. A. MacMahon. (*Proc. Camb. Phil. Soc.* Vol. 20, 1921, pp. 281—283).

"Proof of certain identities and congruences enunciated by S. Ramanujan," by H.B.C. Darling. (*Proc. London Math. Soc.*, Ser. 2, Vol. 19, 1921, pp. 350—372).

"On a type of modular relation," by L. J. Rogers. (*ibid.*, pp. 387—397).

It is plainly impossible for me, within the limits of a notice such as this, to attempt a reasoned estimate of Ramanujan's work. Some of it is very intimately connected with my own, and my verdict could not be impartial; there is much too that I am hardly competent to judge, and there is a mass of unpublished material, in part new and in part anticipated, in part proved and in part only conjectured, that still awaits analysis. But it may be useful, if I state shortly and dogmatically, what seems to me Ramanujan's finest, most independent, and most characteristic work.

His most remarkable papers appear to me to be (3), (7), (9), (17), (18), (19), and (21). The first of these is mainly Indian work, done before he came to England; and much of it had been anticipated. But there is much that is new and in particular a very remarkable series of

algebraic approximations to $\pi$. I may mention only the formulæ

$$\pi = \frac{63}{25} \cdot \frac{17+15\sqrt{5}}{7+15\sqrt{5}}, \quad \frac{1}{2\pi\sqrt{2}} = \frac{1103}{992},$$

correct to 9 and 8 places of decimals respectively.

The long memoir (7) represents work, perhaps, in a backwater of mathematics, and is somewhat overloaded with detail ; but the elementary analysis of "highly composite" numbers—numbers which have more divisors than any preceding number- —is exceedingly remarkable, and shows very clearly Ramanujan's extraordinary mastery over the algebra of inequalities. Papers (9) and (17) should be read together, and in connection with Mr. Mordell's paper mentioned above, for Mr. Mordell afterwards proved a great deal that Mr. Ramanujan conjectured. They contain, in particular, exceedingly remarkable contributions to the theory of the representation of numbers by sums of squares. But I am inclined to think that it was in the theory of partitions, and the allied parts of the theories of elliptic functions and continued fractions, that Ramanujan shone at his very best.

It is in papers (18), (19) and (21), and in the papers of Prof. Rogers and Mr. Darling that I have quoted, that this side of his work (so far as it has been published) is to be found. It would be difficult to find more beautiful formulæ than the "Roger's-Ramanujan" identities, proved in (19) ; but here Ramanujan must take second place to Prof. Rogers ; and if I had to select one formula from all of Ramanujan's work, I would agree with Major MacMahon in selecting a formula from (18), *viz.*

$$p(4) + p(9)x + p(14)x^2 + \ldots\ldots = 5\,\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\ldots\}^5}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^6}$$

where $p\ (n)$ is the number of partitions of $n$.

I have often been asked whether Ramanujan had any special secret ; whether his methods differed in kind from those of other mathematicians ; whether there was anything really abnormal in his mode of thought. I cannot answer these questions with any confidence or conviction ; but I do not believe it. My belief is that all mathematicians think, at bottom, in the same kind of way, and that Ramanujan was no exception. He had,

of course, an extraordinary memory. He could remember the idiosyncracies of numbers in an almost uncanny way. It was Mr. Littlewood (I believe) who remarked that "every positive integer was one of his personal friends." I remember once going to see him when he was lying ill at Putney. I had ridden in Taxi-cab, No. 1729, and remarked that the number (7×13×19) seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No" he replied, "it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways." I asked him, naturally, whether he knew the answer to the corresponding problem for fourth powers; and he replied after a moment's thought, that he could see no obvious example, and thought that the first such number must be very large.* His memory, and his powers of calculation, were very unusual, but they could not reasonably be called "abnormal." If he had to multiply two large numbers, he multiplied them in the ordinary way; he would do it with unusual rapidity and accuracy, but not more rapidly or accurately than any mathematician who is naturally quick and has the habit of computation. There is a table of partitions at the end of our paper (15). This was, for the most part, calculated independently by Ramanujan and Major MacMahon; and Major MacMahon was in general, slightly the quicker and more accurate of the two.

It was his insight into algebraical formulæ, transformations of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler or Jacobi. He worked, far more than the majority of mathematicians, by induction from mumerical examples; all of his congruence properties of partitions, for example, were discovered in this way. But with his memory, his patience, and his power of calculation, he combined a power of generalisation, a feeling for form, and a capacity for rapid modification of his hypotheses, that was often really startling, and made him, in his own peculiar field, without a rival in his day.

It is often said that it is much more difficult now for a mathematician to be original than it was in the great days when the foundations of

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* Euler gave $542^4 + 103^4 = 359^4 + 514^4$ as an example. See Sir T. L. Heath's *Diophantus of Alexandria*, p. 380.

modern analysis were laid, and no doubt in a measure it is true. Opinions may differ as to the importance of Ramanujan's work, the kind of standard by which it should be judged, and the influence which it is likely to have on the mathematics of the future. It has not the simplicity and the inevitableness of the very greatest work, it would be greater if it were less strange. One gift it has which no one can deny, profound and invincible originality. He would probably have been a greater mathematician if he had been caught and tamed a little in his youth ; he would have discovered more that was new, and that, no doubt, of greater importance. On the other hand, he would have been less of a Ramanujan, and more of a European professor, and the loss might have been greater than the gain.

# ON THE DIRECTION ANGLES OF A CIRCLE.

By F. H. V. Gulasekharam, M.A.

*Introductory.* Question 1104 (*J. I. M. S.*) of Mr. Bhimasena Rao suggests that the angles at which a circle cuts the sides of the triangle of reference (ABC) play an important part in certain classes of problems. We may, therefore, call the angles $\alpha$, $\beta$, $\gamma$ at which a circle cuts the sides, BC, CA, AB respectively, "*the direction angles of the circle.*" The circle itself may be called the circle ($\alpha$, $\beta$, $\gamma$). The following equations and properties of a cicle *in terms of its direction angles* may be of interest :—

**1.** *Given the direction angles* ($\alpha$, $\beta$, $\gamma$) *of a circle, to find*

(i) *the normal co-ordinates of its centre* ;

(ii) *an expression for its radius* ;

*and* (iii) *its tangential equation.*

Let $r$ be the radius of the circle, and S, its centre.

(i) The *absolute* normal co-ordinates of the centre are

$$(r \cos \alpha, r \cos \beta, r \cos \gamma.)$$

Hence, the *relative* normal co-ordinates are

$$(\cos \alpha, \cos \beta, \cos \gamma) \quad \ldots \quad \ldots \quad (1.1)$$

(ii) $r(\sin A \cos \alpha + \sin B \cos \beta + \sin C \cos \gamma)$

$= 2R \sin A \sin B \sin C$, where R is the radius of the circumcircle of ABC. ... ... ... (1.2)

(iii) The usual condition for tangency of a circle gives at once the *tangential equation (trilinear)* in the form

$$(l \cos \alpha + m \cos \beta + n \cos \gamma)^2 = \Omega, \quad \ldots \quad (1.3)$$

where $$\Omega = \sum l^2 - 2\sum (mn \cos A). \quad \ldots \quad \ldots \quad (1.4)$$

The equation (1.3) may also take the form

$$\sum l^2 \sin^2 \alpha - 2\sum \{ mn (\cos A + \cos \beta \cos \gamma) \} = 0. \ldots \quad (1.5)$$

Note: We see from the above results that the direction angles *uniquely determine* the circle ($\alpha$, $\beta$, $\gamma$.)

**2.** *To find the condition that the circle* ($\alpha$, $\beta$, $\gamma$) *may be a pedal circle with respect to the triangle of reference* (*ABC*).

Let the circle cut BC, CA, AB in order at the points ($D_1$, $D_2$) ($E_1$, $E_2$), ($F_1$, $F_2$) respectively.

Then $D_1D_2 = 2r \sin \alpha$ ; similarly for $E_2E_1$ and $F_2F_1$.

Also $BD_1 (BD_1 + 2r \sin \alpha) = t_2^2$ ; $CD_1 (CD_1 - 2r \sin \alpha) = t_3^2$.

Hence $BD_1^2 - CD_1^2 + 2r.\, a \sin \alpha = t_2^2 - t_3^2$.

Similarly $CE_1^2 - AE_1^2 + 2r.\, b \sin \beta = t_3^2 - t_1^2$.

And $AF_1^2 - BF_1^2 + 2r.\, c \sin \gamma = t_1^2 - t_2^2$.

Hence for any circle ($\alpha$, $\beta$, $\gamma$),

$$4Rr\,(\text{Sin } A \sin \alpha + \text{Sin } B \sin \beta + \text{Sin } C \sin \gamma)$$
$$= CD_1^2 - BD_1^2 + AE_1^2 - CE_1^2 + BF_1^2 - AF_1^2 \quad \ldots \quad (2.1)$$

But for a pedal circle $D_1E_1F_1$ the R. H. S. of (2.1) vanishes. Hence the necessary and sufficient condition for a pedal circle is given by the relation

$$\Sigma (\sin A \sin \alpha) = 0, \quad \ldots \quad \ldots \quad (2.2)$$

where the angles $\alpha, \beta, \gamma$ as applied to $D_1\ E_1\ F_1$ are to be taken with the proper signs.

Note :—*If* $ul^2 + vm^2 + wn^2 + 2u'mn + 2v'nl + 2w'lm = 0$. *be the tangential equation (trilinear) of a circle, ths condition for a pedal circle is (from* 2.2 *and* 1.5)

$$\sin A.\ \sqrt{u} + \sin B.\ \sqrt{v} + \sin C\ \sqrt{w} = 0 \quad \ldots \quad (2.3)$$

**3.** *To find the condition that the circle* $(\alpha, \beta, \gamma)$ *may be a 'contact circle.'*

Let the circle cut the sides of ABC as before at $(D_1, D_2)$, $(E_1, E_2)$ $(F_1, F_2)$ respectively.

Now if a conic touches the sides of ABC at $D_1$, $E_1$, $F_1$, we have

$$BD_1 . CE_1 . AF_1 = D_1C . E_1A . F_1B. \quad \ldots \quad (3.1)$$

Again projecting $BD_1$ on the perp. from S on AB, we get

$$\left.\begin{aligned} BD_1 . \sin B &= r\,[\cos \gamma + \cos (B + \alpha)]. \\ \text{Similarly } CE_1 . \sin C &= r\,[\cos \alpha + \cos (C + \beta)], \\ AF_1 . \sin A &= r\,[\cos \beta + \cos (A + \gamma)], \\ D_1C . \sin C &= r\,[\cos \beta + \cos (C - \alpha)], \\ E_1A . \sin A &= r\,[\cos \gamma + \cos (A - \beta)], \\ F_1B . \sin B &= r\,[\cos \alpha + \cos (B - \gamma)], \end{aligned}\right\} \quad \ldots \quad (3.2)$$

Hence from (3.1), after cancelling like factors, we get

$$4 \cos \tfrac{1}{2} (B + \gamma + \alpha) \cos \tfrac{1}{2} (C + \alpha + \beta) \cos \tfrac{1}{2} (A + \beta + \gamma)$$
$$= 4 \cos \tfrac{1}{2} (B - \gamma - \alpha) \cos \tfrac{1}{2} (C - \alpha - \beta) \cos \tfrac{1}{2} (A - \beta - \gamma).$$

Hence expressing either side as the sum of four cosines, we get

$$\Sigma [\sin (A - \alpha)] - \sin (\alpha + \beta + \gamma)$$
$$= \Sigma [\sin (A + \alpha)] + \sin (\alpha + \beta + \gamma)$$

whence

$$\cos A \sin \alpha + \cos B \sin \beta + \cos C \sin \gamma + \sin (\alpha + \beta + \gamma) = 0. \quad (3.3)$$

which is the required condition. This may also be written in the form

$$\Sigma (\sin \alpha (\cos A + \cos \beta \cos \gamma)] = \sin \alpha \sin \beta \sin \gamma. \quad \ldots \quad (3.4)$$

where the angles $\alpha, \beta, \gamma$ as applied to $D_1E_1F_1$ are to be taken with the proper signs.

Note :—*If the tangential equation (trilinear) of a circle be* $ul^2 + vm^2 + wn^2 + 2u'mn + 2v'nl + 2w'lm = 0$, *the condition for a contact circle is given by the relation*

$$u' \sqrt{u} + v' \sqrt{v} + w' \sqrt{w} = \sqrt{uvw}.$$

**4.** *To find the angle θ at which the circle (α, β, γ) cuts the circle given by the trilinear equation*

$$f(x, y, z) \equiv (lx + my + nz)\,\Sigma(x \sin A) - k\Sigma(yz . \sin A) = 0.$$

Now if $r'$ be the radius of the latter circle, and $d$ the distance between the two centres, (as in Ex. 2, p. 128, Salmon's *Conic Sections*), we have

$$k . (d^2 - r'^2) \sin A \sin B \sin C = r^2 f(\cos \alpha, \cos \beta, \cos \gamma). \quad (4.1)$$

Again, $$2rr' \cos \theta = r^2 + r'^2 - d^2.$$

$$\therefore\ 2(k . r' \cos \theta)\left(\frac{1}{r}\right) = k - \frac{k}{r^2}(d^2 - r'^2).$$

From (1.2) and (4.1),

$$k + r' \cos \theta . \quad \Sigma \sin A \cos \alpha$$
$$= kR \sin A \sin B \sin C - f(\cos \alpha, \cos \beta, \cos \gamma), \ldots \quad (4.2)$$

where $r'$ is given by the relation

$$k^2 r'^2 = R^2[k^2 - 2k\Sigma l \cos A + \Omega].$$

**5**. *Particular case of* § 4. *N. P. Circle.*

For the N.P. Circle,

$$l = \cos A,\ m = \cos B,\ n = \cos C,\ r' = \frac{R}{2};\ k = 2.$$

Hence from (4 . 2,) *the angle θ at which the circle* (α, β, γ) *cuts the N. P. Circle is given by the equation*

$$\cos\theta \quad \Sigma(\sin A \cos \alpha) = 2 \sin A \sin B \sin C - \Sigma(\sin A \cos \alpha)$$
$$\Sigma(\cos A \cos \alpha) + 2\Sigma \sin A \cos \beta \cos \gamma$$
$$= \Sigma(\sin A \cos A \sin^2 \alpha) + \Sigma(\sin A \cos \beta \cos \gamma). \quad \ldots \quad (5.1)$$

Again the R. H. S. of (5.1) is easily transformed into

$$[\Sigma(\sin A \sin \alpha)][\Sigma(\cos A \sin \alpha)] + \Sigma[\sin A \cos(\beta + \gamma)]$$
$$= [\Sigma(\sin A \sin \alpha)]\,[\Sigma(\cos A \sin \alpha)] + \cos(\alpha + \beta + \gamma)$$
$$[\Sigma(\sin A \cos \alpha)] + \sin(\alpha + \beta + \gamma)[\Sigma(\sin A \sin \alpha)].$$

Hence the equation 5·1 may be written in the form

$$[\cos\theta - \cos(\alpha + \beta + \gamma)]\,[\Sigma(\sin A \cos \alpha)]$$
$$= [\Sigma(\sin A \sin \alpha)]\,[\Sigma(\cos A \sin \alpha) + \sin(\alpha + \beta + \gamma)] \ldots \quad (5.2)$$

**6.** *Solution of Mr. Bhimasena Rao's Q.* 1104.

The equation 4·1 above for cos θ is easily identified with that of Q. 1104.

Again from (2.2), (3.3), and (5.2) for 'pedal circles' and 'contact circles'

$$\theta = \alpha + \beta + \gamma;$$

and conversely.

(*Vide—Post-script*, Mr. Bhimasena Rao's paper : *An Extension of Feuerbach's Theorem*, *J. I. M. S.*, Vol. XI, p. 227. Also Q. 863, of Mr. V. Ramaswami Iyer.]

# SHORT NOTES.

## On the Sine and Cosine Series.

[The following variation of the method of obtaining the Sine and Cosine Series given by Professors Hill and Jackson in the *Messenger*, Vols. 35 & 43 may be of interest as being more direct.]

§ 1. *Lemma I.**

If $U = u_1 - u_2 + u_3 - \ldots\ldots + (-1)^{n-1} u_n + \ldots$
$V = v_1 - v_2 + v_3 - \ldots\ldots \quad (-1)^{n-1} v_n + \ldots$

*be two alternating series whose terms form diminishing sequences, then*

$$UV < u_1 v_1 - (u_1 v_2 + u_2 v_1) + \ldots + (u_1 v_{2n-1} + u_2 v_{2n-2} + \ldots + u_{2n-1} v_1) \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

*where on the right, we take only terms of the product of order not greater than* $2n$; *and*

$$UV > u_1 v_1 - (u_1 v_2 + u_2 v_1) + \ldots - (u_1 v_{2n} + u_2 v_{2n-1} + \ldots + u_{2n} v_1) \qquad \ldots \quad \ldots \quad \ldots \quad (2)$$

*taking on the right terms of order not greater than* $2n+1$.

*Cor.* If $x$ is an acute angle, (1) and (2) can be applied to the series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

and
$$\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots$$

since the terms of both form monotonic diminishing sequences.

*Lemma II. If $f(x)$ is a function such that $f(x) \gtrless \lambda f\left(\frac{x}{\lambda}\right)$, $[\lambda > 1]$*

*and if $\lambda^n f\left(\frac{x}{\lambda^n}\right) \to 0$, as $n \to \infty$, then $f(x) \gtrless 0$ respectively.* (3)

For, by successive reduction, $f(x) > \lambda^n f\left(\frac{x}{\lambda^n}\right)$, however large $n$ may be.

$\therefore \; f(x) > 0$, since $\lambda^n f\left(\frac{x}{\lambda^n}\right) \to 0$, as $n \to \infty$.

*Cor.* Since $2^n \cdot \sin \frac{x}{2^n} \to x$, as $n \to \infty$,

$$\phi(x) \equiv \sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots + (-1)^r \cdot \frac{x^{2r-1}}{2r-1!} \quad (4)$$

is such that $2^n \phi\left(\frac{x}{2^n}\right) \to 0$, as $n \to \infty$.

§ 2. Now $\qquad \sin x < x.$

$$\therefore \qquad 1 - \cos x = 2 \sin^2 \frac{x}{2} < \frac{x^2}{2}.$$

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* This is taken from Prof. Hill's Paper. The proof follows very easily.

Also $2\left(\sin\frac{x}{2}-\frac{x}{2}\right)-(\sin x-x)=2\sin\frac{x}{2}\cdot\left(1-\cos\frac{x}{2}\right)$

$$<\frac{x^3}{2^3}\cdot\frac{1-\frac{1}{2^2}}{1-\frac{1}{2^2}},\quad<\frac{x^3}{3!}-\frac{x^3}{2^2}\cdot\frac{1}{3!}.$$

$$\therefore\quad 2\left(\sin\frac{x}{2}-\frac{x}{2}+\frac{x^3}{2^3}\cdot\frac{1}{3!}\right)<\sin x-x+\frac{x^3}{3!}.$$

$$\therefore\quad \sin x-x+\frac{x^3}{3!}>0.\qquad\text{[by Lemma II.}\qquad(5)$$

Hence
$$\cos x=1-2\sin^2\frac{x}{2}$$
$$<1-2\left(\frac{x}{2}-\frac{x^3}{2^3\ 3!}\right)^2$$
$$<1-\frac{x^2}{2}+\frac{x^4}{4!}.\qquad\text{[by Lemma I.}\qquad(6)$$

Again

$$2\left(\sin\frac{x}{2}-\frac{x}{2}\right)-(\sin x-x)=2\sin\frac{x}{2}\left(1-\cos\frac{x}{2}\right)$$
$$>2\cdot\left(\frac{x}{2}-\frac{x^3}{2^3\ 3!}\right)\left(\frac{x^2}{2^2\ 2!}-\frac{x^4}{2^4\ 4!}\right).$$
$$>\frac{x^3}{2^3}\cdot\frac{1-\frac{1}{2^2}}{1-\frac{1}{2^2}}-3\cdot\frac{x^5}{2^4\ 4!}\cdot\frac{1-\frac{1}{2^4}}{1-\frac{1}{2^4}}.$$
$$>\frac{x^3}{3!}-\frac{x^3}{2^2}\cdot\frac{1}{3!}-\frac{x^5}{5!}+\frac{x^5}{2^4}\cdot\frac{1}{5!}.$$

$$\therefore\ 2\left(\sin\frac{x}{2}-\frac{x}{2}+\frac{x^3}{2^3\ 3!}-\frac{x^5}{2^5\ 5!}\right)>\sin x-x+\frac{x^3}{3!}-\frac{x^5}{5!}$$

$$\therefore\quad \sin x-x+\frac{x^3}{3!}-\frac{x^5}{5!}<0.\qquad\text{[by Lemma II.}\qquad(7)$$

§ 3. To complete the proof by induction, assume that

$$\sin x<x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots+\frac{x^{4r+1}}{\overline{4r+1}!},$$
$$\cos x>1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots+\frac{x^{4r}}{4r!}-\frac{x^{4r+2}}{\overline{4r+2}!}.$$

Now $2\left(\sin\frac{x}{2}-\frac{x}{2}\right)-\sin x)-x)=2\sin\frac{x}{2}\left(1-\cos\frac{x}{2}\right)$

$$>2\left(\frac{x}{2}-\frac{x^3}{2^3\,3!}+\dots+\frac{x^{4r+1}}{2^{4r+1}\ \overline{4r+1}!}\right)\times$$
$$\left(\frac{x^2}{2^2\,2!}-\frac{x^4}{2^4\ 4!}+\dots-\frac{x^{4r+2}}{2^{4r+2}\ \overline{4r+2}!}\right).$$

By (1) and (2) above, we need only take the terms up to and including $x^{4r+3}$. And the co-efficient of $x^{4r+3}$ in the product is negative and

$$= \frac{1}{2^{4r+2}} \cdot \left\{ \frac{1}{1!\,\underline{4r+2}!} + \frac{1}{3!\,4r!} + \ldots + \frac{1}{\underline{4r+1}!\,2!} \right\}$$

$$= \frac{1}{2^{4r+2}} \cdot \frac{1}{\underline{4r+3}!} \left\{ \frac{1}{2}(1+1)^{4r+3} - 1 \right\}$$

$$= \frac{1}{\underline{4r+3}!} - \frac{1}{2^{4r+2}} \cdot \frac{1}{\underline{4r+3}!}.$$

$$\therefore \text{ As in (7), } \quad \sin x - x + \frac{x^3}{3!} - \ldots - \frac{x^{4r+1}}{\underline{4r+1}!} + \frac{x^{4r+3}}{\underline{4r+3}!} > 0.$$

Applying this in $\cos x = 1 - 2\sin^2 \frac{x}{2}$, we get as in (6) after a similar simple reduction,

$$\cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{x^{4r+4}}{\underline{4r+4}!}.$$

Hence the usual series.

§ 4. The identity $3 \sin \frac{x}{3} - \sin x = 4 \sin^3 \frac{x}{3}$ can be similarly employed to establish the series.

C. KRISHNAMACHARI.

# SOLUTIONS.

## Question 1137.

(Martyn M. Thomas) :—Two particles moving with velocities 1 : $k$ are at corresponding points of a curve and its $n$th negative pedal. Show that the differential equation of the original curve is

$$\frac{dr}{d\theta} = r^{1-1/n} \sqrt{(kr+a)^{2/n} - r^{2/n}}$$

where $a$ is an arbitrary constant.

Each radius vector of the curve $r = b\cos(\theta\sin\alpha)$ is diminished by $b\cos\alpha$. Show that this new curve and its negative pedal can be described by two particles simultaneously with velocities $\sec\alpha : 1$.

*Solution by K. Satyanarayana.*

If P,Q be two consecutive points on a curve $p = f(r)$, and P, Q′ the feet of perpendiculars from the origin on the tangents thereat, it is easy to see that $P'Q' = rd\psi$, in the limit;

i.e. $$ds' = rd\psi = \frac{r}{\rho}\,ds.$$

∴ if $p_{n-1},\ r_{n-1}$ and $p_n,\ r_n$ represent co-ordinates for the $(n-1)^{th}$ and $n^{th}$ negative pedal of $p = f(r)$; then

$$ds_n = \frac{\rho_n}{r_n}\,ds_{n-1}.$$

$$\therefore\quad ds_n = \frac{\rho_n\,.\,\rho_{n-1}\,.\,\rho_{n-2}\cdots\rho_1}{r_n\,.\,r_{n-1}\cdots\cdots\cdots r_1}\,ds\ . = \frac{dr_1}{dp_1}\,.\,\frac{dr_2}{dp_2}\,.\,\frac{dr_3}{dp_3}\cdots\frac{dr_n}{dp_n}\,.\,ds.$$

i.e. $$\frac{ds_n}{ds} = \frac{dr_1}{dp_1}\,.\,\frac{dr_2}{dp^2}\cdots\frac{dr_n}{dp^n}\ . = k, \text{ by the problem.}$$

Now $$p_n = r_{n-1} \text{ and } \frac{r_n}{p_n} = \frac{r}{p}.$$

$$\therefore\ \frac{r_n}{r} = \frac{r_n}{r_{n\ldots1}}\,.\,\frac{r_{n\ldots1}}{r_{n\ldots2}}\cdots\frac{r_1}{r} = \frac{r_n}{p_n}\,.\,\frac{r_{n\ldots1}}{p_{n\ldots1}}\cdots\frac{r_1}{p_1} = \left(\frac{r}{p}\right)^n.$$

$$\therefore\qquad r_n = \frac{r_{n+1}}{p^n}$$

$$\therefore\qquad \frac{dr_n}{dp_n} = \frac{dr_n}{dp} \div \frac{dr_{n-1}}{dp} = \frac{r}{p}\,.\,\frac{(n+1)\,p\rho - nr^2}{n\,p\rho - (n-1)r^2}.$$

$$\therefore\qquad \frac{dr_n}{dp_n}\,.\,\frac{dr_{n-1}}{dp_{n-1}}\cdots\frac{dr_1}{dp_1} = \left(\frac{r}{p}\right)^n.\ \frac{(n+1)\,p\rho - nr^2}{p\rho} = k,$$

Hence we have the differential equation to the path, namely,

$$n\,.\,\frac{r^2}{p\rho} = \left[(n+1) - k\ \left(\frac{p}{r}\right)^n\right]$$

or $$n \cdot \frac{dp}{dr} = \left[(n+1) - k \left(\frac{p}{r}\right)^n\right] \frac{p}{r}.$$

To solve this put $\frac{p}{r} = \lambda$ and the equation reduces to

$$n\left[r \cdot \frac{d\lambda}{dr} + \lambda\right] = \left[(n+1) - k\,\lambda^n\right]\lambda$$

or $$nr \cdot \frac{d\lambda}{dr} = (1 - k\lambda^n)\,\lambda,$$

or $$\frac{n\,d\lambda}{\lambda\,(1 - k\lambda^n)} = \frac{dr}{r},$$

whose solution is $$\frac{k\lambda^n}{1 - k\lambda^n} = ar,$$

or $$k\lambda^n = \frac{ar}{1 + ar},$$

where $a$ is an arbitrary constant.

Hence $$\left(\frac{p}{r}\right)^n = \frac{ar}{(1 + ar)k},$$

or $$\left(r\frac{d\theta}{dr}\right)^2 = \frac{(ar)^{\frac{2}{n}}}{\{k(1 + ar)\}^{\frac{2}{n}} - (ar)^{\frac{2}{n}}}$$

whence the result stated in the question follows.

For the second part $n = 1$, $k = \cos\alpha$; accordingly the curve is

$$\frac{dr}{d\theta} = \sqrt{(a + r\cos\alpha)^2 - r^2}.$$

$$\therefore \quad \frac{dr}{\sqrt{(a\operatorname{cosec}\alpha)^2 - (r\sin\alpha - a\cot\alpha)^2}} = d\theta,$$

giving $$\frac{r\sin\alpha - a\cot\alpha}{a\operatorname{cosec}\alpha} = \sin[\theta\sin\alpha + \beta].$$

Putting $\beta = \frac{\pi}{2}$ the equation becomes

$$r\sin\alpha = a\cot\alpha + a\operatorname{cosec}\alpha\cos(\theta\sin\alpha).$$

$$\therefore \quad r = \frac{a}{\sin^2\alpha} \cdot \cos\alpha + \frac{a}{\sin^2\alpha}\cos(\theta\sin\alpha)$$

$$= b\cos\alpha + b\cos(\theta\sin\alpha)$$

or $$(r - b\cos\alpha) = b\cos(\theta\sin\alpha)$$

which is a curve got by *increasing* by $b\cos\alpha$ each radius vector of $r = b\cos(\theta\sin\alpha)$, and not by *diminishing* as stated in the question.

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**Question 1142.**

(K. ANANDA RAO) :—Let $\omega(n)$ and $\mu(n)$ be defined as follows :

$\omega(n) = 0$, if $n$ has any prime factor repeated more than twice,
$= (-1)^r\, 2^r$ otherwise, $r$ being the number of non-repeated prime factors of $n$ ;

$\mu(n) = 0$, if $n$ has any repeated prime factors
$= (-1)^r$ otherwise, $r$ having the same meaning as above.

Show that for $|x| < 1$

$$\sum_{n=1}^{\infty} \mu(n)\, x^n = \sum_{n=1}^{\infty} \frac{\omega(n)\, x^n}{1 - x^n}.$$

*Solution by S. Purushotham.*

To prove that

$$\sum_{1}^{\infty} \frac{\mu(n)\, x^n}{1 - x^n} = \sum_{1}^{\infty} \omega(n) \left\{ x^n + x^{2n} + x^{3n} + \dots \text{ to } \infty \right\}.$$

Case (1). Let $N = a \times b \times c \dots$ to $n$ factors, where $a, b, c \dots$ are all prime numbers and there are no repeated factors.

Then the coefficient of $x^N$ on R. H. S. is

$$\omega(1)\ \{\omega(a) + \omega(b) + \dots\} + \{\omega(ab) + \omega(bc) + \dots\} + \omega\{abc \dots\}$$
$$= 1 + \{(-2) + (-2) + \dots \text{ to } c_1 \text{ terms}\} + \{(-2)^2 + (-2)^2 + \dots \text{ to } c_2 \text{ terms}\} + \dots\dots + (-2)^n$$
$$= 1 + c_1(-2) + c_2(-2)^2 \qquad + c_n(-2)^n$$
$$= (1-2)^n = (-1)^n = \mu(N) \text{ in this case.}$$

Case (2) Let $N = a \times b \times c \times \dots \times p^{k_1} \times q^{k_2} \times r^{k_3} \dots$

where $a, b, c \dots$ are non-repeated prime numbers and $p, q, r \dots$ are the repeated prime numbers.

Let the number of quantities $a, b, c, \dots p, q, r \dots$ be $n$ and let the number of quantities $p, q, r \dots$ be $m$ ; then

the coefficient of $x^N$ on R. H. S. is

$$[\omega(1) + \{\omega(a) + \omega(b) + \ldots + \omega(p) + \ldots\} + \{\omega(ab) + \omega(ap) + \ldots\} + \ldots] + [\{\omega(ap^2) + \omega(qp^2) + \ldots\} \ldots] + [\{\omega(ap^2 q^2) + \omega(rp^2q^2) + \ldots\} \ldots] + \ldots$$

The sum of terms in the 1st rectangular brackets is $(-1)^n$ as in case (1); and

$$\omega(ap^2) + \omega(qp^2) + \ldots + \omega(ab\,p^2) + \omega(aq\,p^2) + \ldots$$
$$= \omega(a) + \omega(b) + \ldots + \omega(q) \quad + \omega(ab) + \omega(aq) + \ldots$$

$p$ being omitted

$$= (-1)^{n-1}.$$

Similarly for terms involving $q^2$, $r^2$...

Hence the $\omega$'s involving a single square factor give $(-1)^{n-1}\,{}_mC_1$

Similarly $\omega$'s involving $p^2q^2$, $p^2r^2$, etc. give $(-1)^{n-2}\,{}_mC_2$;

and so on.

Thus the total sum $= (-1)^n + (-1)^{n-1}\,{}_mC_1 + (-1)^{n-2}\,{}_mC_2 + \ldots + (-1)^{n-m}\,{}_mC_m$
$= 0 = \mu(N)$ in this case also.

Hence the identity.

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## Question 1153.

(R. Vaidyanathaswami):—Find three collinear points on a cusped cubic the tangents at which are concurrent. Prove that the line containing the three points passes through a fixed point and the point of concurrence of the tangents traces a fixed line.

*Solution by C. N. Sreenivasa Iyengar.*

The equation of any cuspidal cubic can always be reduced to the standard form $zy^2 = x^3$, and any point on this is represented parametrically by $(t, 1, t^3)$. If the three points $t_1, t_2, t_3$ lie on $lx + my + nz = 0$, then $t_1, t_2, t_3$ are the roots of $lt + m + nt^3 = 0$. Hence the necessary and sufficient condition for collinearity is $t_1 + t_2 + t_3 = 0$. (1).

Equation of the tangent at '$t$' is $-x \cdot 3t^2 + y \cdot 2t^3 + z = 0$.

Hence the tangents at $t_1, t_2, t_3$ are concurrent, if

$$\begin{vmatrix} t_1^3 & t_1^2 & 1 \\ t_2^3 & t_2^2 & 1 \\ t_3^3 & t_3^2 & 1 \end{vmatrix} = 0,$$

*i.e.*, if $(t_1 - t_2)(t_2 - t_3)(t - t_1)(t_1t_2 + t_2t_3 + t_3t_1) = 0$,

*i.e.*, if $t_1t_2 + t_2t_3 + t_3t_1 = 0$ (2), since $t_1, t_2, t_3$ are unequal.

By (1) & (2), it follows that the $t$'s are the roots of $t^3 = k$, where $k$ is any constant.

$$\therefore (t_1, t_2, t_3) = (k^{\frac{1}{3}}, \omega k^{\frac{1}{3}}, \omega^2 k^{\frac{1}{3}}),$$

$\omega$ being an imaginary cube root of unity.

The line joining $t_1$ and $t_2$ is

$$\begin{vmatrix} x & y & z \\ k^{\frac{1}{3}} & 1 & k \\ k^{\frac{1}{3}}\omega & 1 & k \end{vmatrix} = 0,$$

which reduces to the form $Ay + Bz = 0$. This always passes through the vertex A of the triangle of reference, *i.e.*, through the intersection of the cuspidal and inflexional tangents of the cubic. This proves the first part of the question.

The tangents at $t_1 = k^{\frac{1}{3}}$, and $t_2 = k^{\frac{1}{3}}\omega$, are respectively

$$\left. \begin{aligned} -x \cdot 3k^{2/3} \quad &+ 2ky + z = 0 \\ -x \cdot 3k^{2/3}\omega^2 &+ 2ky + z = 0 \end{aligned} \right\}.$$

The point of concurrence of the tangents is at the point given by $x = 0$; $2ky + z = 0$. This point always lie on $x = 0$, for all values of $k$. Thus the point of concurrence traces the line $x = 0$, which is the line joining the cusp and the inflexion. Hence the second part.

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## Question 1154.

(R. Vaidyanathaswami) :—Find all the conics which reciprocate a cusped cubic into itself.

*Solution by C. N. Sreenivasa Iyengar.*

The standard equation of a cuspidal cubic is $zy^2 = x^3$. The tangent at $(t, 1, t^3)$ to this is $-x \cdot 3t^2 + y \cdot 2t^3 + z = 0$. The co-ordinates $(x_1, y_1, z)$ of the pole of this line *w.r.t.* $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, are given by

$$\frac{ax_1 + hy_1 + gz_1}{-3t^2} = \frac{hx_1 + by_1 + fz_1}{2t^3} = \frac{gx_1 + fy_1 + cz_1}{1}.$$

Eliminating $t$, the locus of $(x_1, y_1, z_1)$, or the required polar reciprocal, is found to be

$$\frac{(ax + hy + gz)^3}{-27} = \frac{(hx + by + fz)^2 (gx + fy + cz)}{4}.$$

If this should be identical with $zy^2 = x^3$,

$$f = g = h = 0; \text{ and } \frac{a^3}{27} + \frac{b^2c}{4} = 0.$$

The equation of the conic thus becomes $ax^2 + by^2 + cz^2 = 0$, where $a, b, c$ are connected by the relation $\frac{a^3}{27} + \frac{b^2c}{4} = 0.$

There exist therefore a single infinity of conics, all of which are self-conjugate *w.r.t.* the $\Delta$ of reference, *i.e.*, the triangle formed by the cuspidal tangent, inflexional tangent, and the line joining the cusp and inflexion.

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## Question 1162.

(K. J. Sanjana):—Solve the following differential equations and explain their geometrical significance:—

(i) $x^2+y^2-2\frac{xy_1(1+y_1^2)}{y_2}+\frac{2y(1+y_1^2)}{y_2}=k^2$;

(ii) $x^2+y^2-2\frac{xy_1(1+y_1^2)}{y_2}+\frac{2y(1+y_1^2)}{y_2}+\frac{(1+y_1^2)^3}{y_2^2}=k^2$;

(iii) $y-mx+\frac{1+y_1^2}{y_2}+\frac{m\,y_1(1+y_1^2)}{y_2}=0$,

where $y_1$ and $y_2$ stand for $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ and $k$ and $m$ are given constants.

*Solution by C.N. Sreenivasa Iyengar.*

If $(x, y)$ denote the co-ordinates of a point on a given plane curve then the centre of curvature has for co-ordinates: $\frac{y_1(1+y_1^2)}{y_2}, -\frac{(1+y_1^2)}{y_2}$

Equation (i) therefore expresses the fact that every point of the given curve lies on a circle drawn with the corresponding centre of curvature and having a radius equal to $\sqrt{\frac{(1+y_1^2)^3}{y_2^2}+k^2}$. This circle must necessarily coincide with the circle of curvature. Hence

$$\rho^2 = \frac{(1+y_1^2)^3}{y_2^2} + k^2,$$

which is possible only if $\rho = \infty$. The primitive therefore consists of the general equation of the st. line: $y=mx+c$, where $m$ and $c$ are arbitrary. Verification is easily made.

Differential equation (ii) expresses the fact that every point of the given curve lies on a circle drawn with its centre at the corresponding centre of curvature, having its radius equal to $\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$ and *further passing through the origin.* This circle must necessarily coincide with the corresponding circle of curvature, and since it always passes through the origin, the solution consists of the general equation of a system of circles passing through the origin. The general primitive is therefore $x^2 + y^2 + \mathrm{A}x + \mathrm{B}y = 0$, where A and B are arbitrary.

Equation (iii) expresses the fact that every point of the given curve lies on a line drawn in a fixed direction (of slope $m$) through the corresponding centre of curvature. The solution therefore consists of a system of straight lines drawn in a given direction ; viz., *at right angles* to the above ; and therefore represented by

$$x + my = c$$

where $m$ is the same constant as above, and $c$ is arbitrary.

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### Question 1180.

(F. H. V. Gulasekharam) :—If $g$ and $h$ are the lengths of the bisectors of the angles B and C respectively of a triangle ABC, prove that

$$4a^2 > g^2 + h^2.$$

*Solution by A. A. Krishnaswamiengar and others.*

Let the bisectors of the angles B and C meet in I. Then since $\mathrm{B\hat{I}C} = 90^0 + \frac{\mathrm{A}}{2}$, it is an obtuse angle, so that in the triangle BIC,

$$\mathrm{BC}^2 > \mathrm{BI}^2 + \mathrm{IC}^2.$$

Now it is obvious that $\mathrm{BI} > \frac{g}{2}$ and $\mathrm{IC} > \frac{h}{2}$, for if the bisectors BI CI meet AC, AB in D and E respectively

$$\frac{\mathrm{BI}}{\mathrm{ID}} = \frac{\mathrm{AB} + \mathrm{BC}}{\mathrm{AC}} > 1 \text{ ; and similarly } \frac{\mathrm{CI}}{\mathrm{ID}} > 1.$$

Hence it follows that

$$a^2 > \left(\frac{g}{2}\right)^2 + \left(\frac{h}{2}\right)^2,$$

*i.e.*
$$4a^2 > g^2 + h^2.$$

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### Question 1181.

(C. N. Sreenivasa Aiyangar) :—A′B′C′ is the triangle formed by joining the mid. points of $\triangle$ ABC ; A″B″C″ is similarly formed from

A′B′C′, and so on. If S, S′, S″, etc., N, N′, N″, etc., and O, O′, O″, etc., denote the corresponding circum-centres, nine-point centres and ortho-centres respectively, then the three sets of points all tend to the same limit. Find the exact position of this limit in relation to the triangle ABC.

*Solution by A. A. Krishnaswami Iyengar, A. Mahalingam and K. Satyanarayana.*

We know that the nine-point centre of the △ ABC is the circum-centre of the triangle A′B′C′ and also the ortho-centre of the triangle A″B″C″. Hence the three sets of points all tend to the same limit. Further, since all the triangles ABC, A′B′C′, A″B″C″, ...... etc., have a common centroid G which divides the straight line joining the circum- and ortho-centre of each triangle in the same invariable ratio 1 : 2, it follows that when the circum-centre and ortho-centre coincide in the limit, all the intermediate points coincide with the same limiting point which is obviously G, since G is one such intermediate point whose limiting position is itself.

It is, however, interesting to remark that the circum-centre, the nine-point centre and the ortho-centre change their positions alternately from one side to the other of the limiting point G, with an amplitude which is continuously decreasing in the ratio of 2 : 1 and which ultimately vanishes.

## QUESTIONS FOR SOLUTION.

1224. (A. A. KRISHNASWAMI AIYANGAR) :—If H be the in-Feurbach point and $x$, $y$, $z$, the middle points of the sides AB, BC, CA of a triangle ABC, prove that, with a suitable modification of signs,

$$HX + HY + HZ = 0.$$

1225. (B. B. BAGI) :—ABCD is a quadrilateral in which the diagona BD, AC, intersect in O at right angles. OP, OQ, OR, OS are drawn perpendicular to AB, BC, CD, DA meeting them in P, Q, R, S, and the sides CD, DA, AB, BC in P′, Q′, R′, S′. Shew that the eight points PQRS P′Q′R′S′ are concyclic.

1226. (B. B. BAGI) :—ABC, DEF are six points on a conic. DE, DF meet BC in $L_1$, $L_2$; EF, ED meet CA in $M_1$, $M_2$; and FD, FE meet AB in $N_1$, $N_2$; then if AD, BE, CF meet in a point O, then $L_1N_2$, $N_1M_2$, $M_1L_2$ meet in the same point O.

1227. (Prof. K. J. SANJANA) :—A heavy homogeneous string is tightly wrapped round a plane vertical curve which is such that the pressure of the curve per unit length varies as the weight of unit length of the string: prove that the intrinsic equation of the curve is

$$s = C\,l - \frac{l \sin\phi}{(k-1)(k+\cos\phi)} + \frac{kl}{(k^2-1)^{\frac{3}{2}}} \cdot \cos^{-1}\frac{1 + k\cos\phi}{k + \cos\phi},$$

where $k$ is the constant of variation and $l$ stands for $k\,t_0 \div mg$, $m$ being the mass of the string per unit length and $t_0$ the tension at the point where the tangent of the curve is vertical. [When $k$ is $<1$, the last term contains the inverse cosh.]

When the pressure is *equal* to the weight, prove that

$$s = \tfrac{1}{6}\,l(C + 3\tan\tfrac{1}{2}\phi + \tan^3\tfrac{1}{2}\phi),$$

and shew that the curve is then the first negative pedal of a parabola with respect to the focus.

1228. (Prof. K. J. SANJANA) :—Given the intrinsic equation of a curve, shew how to find the intrinsic equations of its first positive and negative pedals.

1229. (Prof. K. J. SANJANA) :—Shew how to find the equation of the surface traced out by the shortest distances of all points of a given straight line from a given conic section whose plane does not contain the straight line, and obtain the result when the given conic is a circle.

1230. (HEMRAJ) :—If the feet of the normals from the focus-acnode of a cubic form an equilateral triangle, prove that the normals are equally inclined to one another or one bisects the angle between the other two.

1231. (HEMRAJ):—Show that the duplication of a cube can be effected by means of either curve given by

$$x^3 = a(x^2 + y^2) \text{ and } x^3 + y^3 = ay^2.$$

Show further how to construct the 1st curve and obtain graphically the side of a cube whose volume is $n$-times that of a given cube.

Establish the following :—

$$\frac{19}{40} - \frac{2\pi}{9\sqrt{3}} = \sum_{1}^{\infty} \frac{2 \cdot 5 \cdot 8 \ldots\ldots (3n-1)}{3^n (3n+8)(n+1)!}.$$

1232. [G. V. T.]:—The perpendiculars from A, B, C on the sides BC, CA, AB of a triangle ABC are divided at L, M, N in the ratio of $k : 1$. O and H are the circumcentre and the orthocentre of the triangle. OL, OM, ON meet BC, CA, AB in X, Y, Z respectively. Show that AX, BY, CZ meet at a point on OH. If L′, M′, N′ be the isogonal conjugates of L, M, N with respect to the triangle ABC, show that XL′, YM′, ZN′ intersect at H.

1232. [G. V. T.]:—P, P′; Q, Q′; R, R′ are opposite vertices of a complete quadrilateral circumscribing a circle concentric with the in-circle of a triangle. If P, P′ are isogonal conjugates with respect to the triangle, show that Q, Q′; R, R′ are also isogonal conjugates.

1234. (M. VAIDYANATHAN, M.A.):—$2^n$ players of equal skill enter for a tournament. They are drawn in pairs and the winners of the first round are drawn again for the next and so on. Find (1) the probability that two given competitors will play against each other in the course of the tournament, and (2) the probability that a given player will win.

1235. (M. VAIDYANATHAN, M A.):—Three events A, B, C are known to have happened in the same century. What is the chance that the events happened in the order A, B, C; B happening within $n$ years of the middle of the century?

1236. (Vidvan G. SOMANNA, M.A , L.T.):—The sum of four different weights is 40 seers. The weights can be manipulated to deal out any quantity weighing up to 40 seers, in integral seers. Find the weights.

1237. (Vidvan G. SOMANNA, M.A., L T.):—The aggregate value of ten different coins is Rs. 360. If they can be manipulated to pay any sum (in integral rupees) up to Rs. 360, find the values of the coins.